Transmittance of light through air-water boundaries at different optical depths- A correction to the Lambert Beer Equation

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Introduction:

In our paper, we have analyzed the path lengths of light waves as they travel through air-water boundaries. The situation we modelled was that of light being scattered by the clouds and hitting the surface of a still pond at various angles and reaching a certain point below the water surface.

We have used Lambert Beer's equation to first model this situation and then we modelled the same situation using our equation. Our equation takes in to account the refraction of light as it passes the air-water boundary, and makes use of the rates of change of incident and refractive angles. The objective of our paper is to test the accuracy of our model and compute the error factor between the two model (if any).

The Problem:

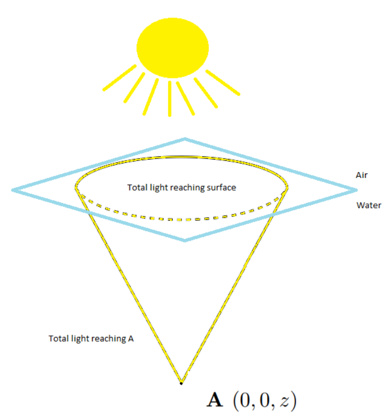

Figure 1: Illustration of problem

The problem is to calculate the total light reaching a certain point below the water surface on a cloudy day. The point of evaluation was taken to be an arbitrary point A with position:

$$(x, y, z) = (0, 0, z)$$

Modelling assumptions used throughout both the models were as follows:

- There was no wave motion on the surface of the water
- The sun was vertically above point A in the sky
- The temperature, density, and pressure of both air and water was assumed to be constant throughout
- The refractive index of water, n_w was taken to be constant at all depths and was assumed to be 1.334
- The optical density of water was assumed to constant at all depths
- The light hitting the surface of the water was randomly scattered by clouds

Lambert Beer Model:

According to Lambert Beer law, light reaching a certain point in an optically dense medium is modelled by the following equation:

$$p = p_0 e^{an(z-z_0)}$$

where, p = number of photons per area, n = number of atoms per volume of media of transmission, a = is the effective area of each atoms and z = depth of point that light reaches in the media.

For our problem, the Lambert Beer equation can be simplified into:

$$p = e^{-z} \tag{1}$$

Equation(1) calculates the total amount of light reaching point A.

Our Model:

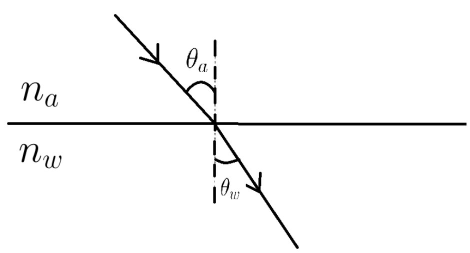

Figure 2: Refraction of light

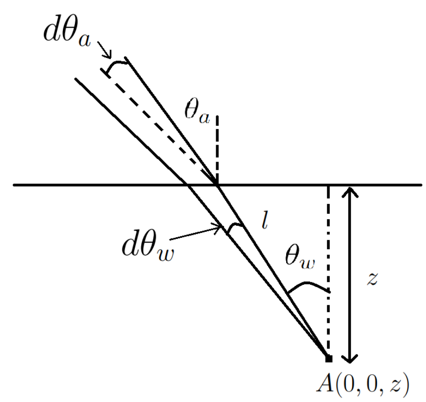

Figure 3: Rate of change of $d\theta_a$ with respect to $d\theta_w$

From Figure(3), we can write the path length, l, of the light ray as a function of z and θ_w:

$$l = \frac{z}{\cos\theta_w} \tag{2}$$

Let S_a , be the light incident on the water surface per unit rad. Therefore, the total light hitting water at any point is given by

$$\int_{-\pi/2}^{\pi/2} S_a d\theta_a = \pi S_a \tag{3}$$

Accounting for the refraction of the incident light rays and integrating over the region, where θ_w is between $(\theta_{max}, -\theta_{max})$ where θ_w is measured from the normal. We get Equation(4), where $R(l)$ represent the total light reaching point A.

$$R(l) = \int_{-\theta_{max}}^{\theta_{max}} e^{kl} \frac{d\theta_a}{d\theta_w} d\theta_w \tag{4}$$

Subsitituting l from Equation(2) into Equation(3), we get Equation(5):

$$R(z) = \int_{-\theta_{max}}^{\theta_{max}} e^{\frac{-kz}{\cos\theta_w}} \frac{d\theta_a}{d\theta_w} d\theta_w \tag{5}$$

In our modelling we have made extensive use of Snell's law:

$$n_a \sin\theta_a = n_w \sin\theta_w \tag{6}$$

Taking the derivative of Snell's law with respect to θ:

$$n_a \cos\theta_a d\theta_a = n_w \cos\theta_w d\theta_w$$

$$\Rightarrow \frac{d\theta_a}{d\theta_w} = \frac{n_w}{n_a} \frac{\cos\theta_w}{\cos\theta_a}$$

From Equation(6), we can write $\cos\theta_a$ as:

$$\cos\theta_a = \sqrt{1 - (\frac{n_w}{n_a})^2 (\sin^2\theta_w)}$$

$$\therefore \frac{d\theta_a}{d\theta_w} = \frac{n_w}{n_a} \frac{\cos\theta_w}{\sqrt{1 - (\frac{n_w}{n_a})^2 \sin^2\theta_w}} \tag{7}$$

At θ_{max}, the angle of incidence, $\theta_a = \frac{\Pi}{2}$, which means that $\sin\theta_a = 1$. We also know that the refraction index of water, $n_w = 1.334$ and air, $n_a = 1$.

$$\theta_{max} = \sin^{-1}(\frac{n_a}{n_w} \sin\theta_a)$$

$$n_0 = \frac{n_w}{n_a} = 1.334 \tag{8}$$

$$\Rightarrow \theta_{max} = \sin^{-1}(\frac{n_a}{n_w}) = \sin^{-1}(\frac{1}{n_0})$$

$$\therefore \theta_{max} = \sin^{-1}(\frac{1}{1.334})$$

Combining Equations (7) and (8) into Equation(5), we get Equation(9):

$$R(z) = \int_{-\theta_{max}}^{\theta_{max}} \frac{n_0 e^{-\frac{kz}{\cos\theta_w}} \cos\theta_w}{\sqrt{1 - (n_0 \sin\theta_w)^2}} d\theta \tag{9}$$

Let the quantity kz be ξ which represents optical depth. Equation(10) is, thus, a function of optical depth and refractive index that calculates the total amount of light reaching point A in 2 dimensions.

$$R(\xi, n_0) = \int_{-\theta_{max}}^{\theta_{max}} \frac{n_0 e^{-\frac{\xi}{\cos\theta_w}} \cos\theta_w}{\sqrt{1-(n_0 \sin\theta_w)^2}} d\theta \tag{10}$$

Equation(11) can be derived from Equation(10) by taking the integral of $R(\xi, n_0)$ over a circular area on the surface of the water and using the polar jacobian. The integrand in Equation(11) models the path lengths of light rays reaching point A in 3 dimensions.

$$R(\xi, n_0) = \int_{-\theta_{max}}^{\theta_{max}} \frac{n_0 \pi \sin\theta_w \cos\theta_w}{\sqrt{1-(n_0 \sin\theta_w)^2}} e^{-\frac{\xi}{\cos\theta_w}} d\theta \tag{11}$$

Equation (11) is the equation we use to calculate the total amount of light that reaches point A.

Analysis

In this section we compare the Equation(11) to the Lambert Beer Equation (Equation (1)).

We plotted the integrand from Equation(11) with respect against varying values for θ_{max} from 0 to 2π at different optical depths.

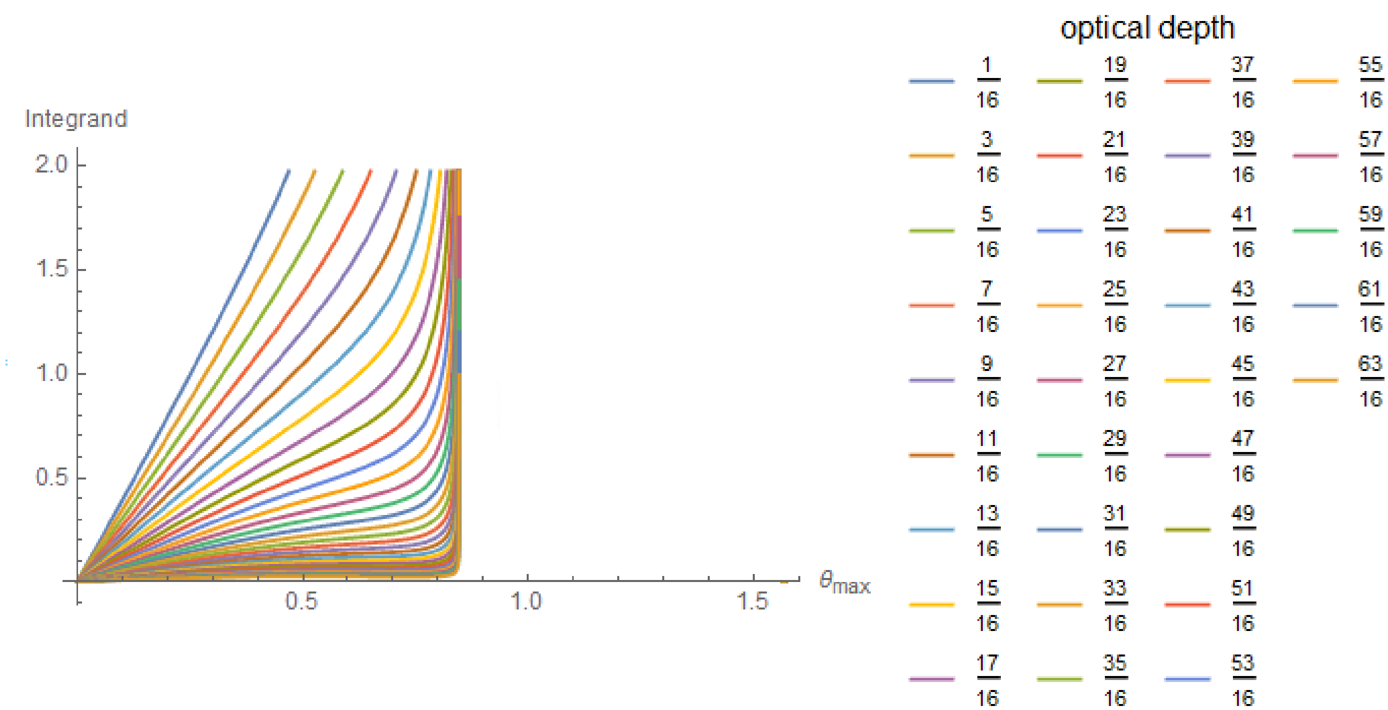

Figure 4: Plot of the integrand at different optical depths against various θ_{max}

The figure shows how the path lengths of the light changes with different θ_{max} values and at different optical depths, z. The graph is discontinuous around $\theta_{max} = \arcsin(\frac{1}{1.334})$ just as we predicted and the path lengths of light become exponentially shorter at higher optical depths.

At optical depth= 0, $R(0, n_0) = 0.749625$. Using this result, we can normalize Equation(11) into Equation(12):

$$\frac{R(\xi, n_0)}{R(0, n_0)} = \frac{\int_{-\theta_{max}}^{\theta_{max}} \frac{n_0 e^{-\frac{\xi}{\cos\theta_w}} \cos\theta_w}{\sqrt{1-(n_0 \sin\theta_w)^2}} d\theta}{0.749625} \tag{12}$$

Plotting Equation(12) and Lambert Beer Equation (Equation(1)) against optical depths, $\xi(0.01, 10)$ by using a log-log plot we get Figure(5):

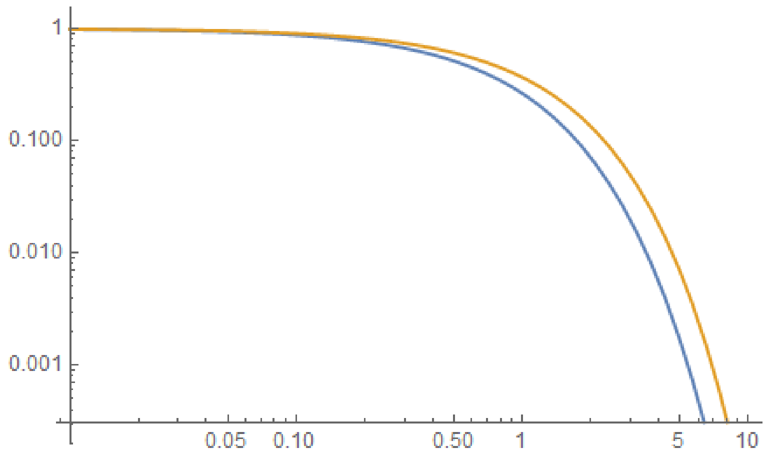

Figure 5: Log-log Plot of $\frac{R(\xi,n_0)}{R(0,n_0)}$ (blue curve) and e^{-z} (yellow curve) against ξ from 0.01 to 10

From Figure(5), we can see that both the equations agree till about optical depth of 0.10. At higher optical depths, there is a significant difference between the two equations. This difference is due to the accounting for refraction of light in Equation(11) and not in Equation(1), making Equation(11) and our model more accurate than that of Lambert Beer's.

In order to measure this error in the Lambert Beer Model we plot the ratio of Equation(11) and Equation(1) against varying optical depths, $\xi(0.01, 10)$.

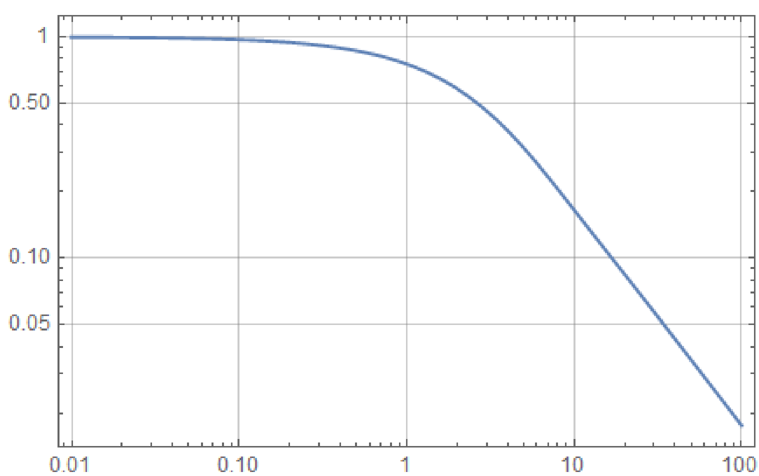

Figure 6: Log-log Plot of $\frac{R(\xi,n_0)}{R(0,n_0)}/e^{-z}$ against ξ from 0.01 to 100

From Figure(6), we can see that at optical depths greater than 0.1 , the Lambert Beer equation starts to differ greatly from our Equation(11). At around $\xi = 7$ Equation 1 yields results that differ from ours by a factor of approximately 2. The error rate increases even more rapidly at higher optical depths.

Conclusion:

Our analysis has shown that our Equation(10) is more accurate than Equation(1) at higher optical depths. It can be said that the Lambert-Beer Equation is an approximation to our equation at lower optical depths. It has also proven that taking into the refraction of light significantly affects the results at higher optical depths. Our equation, thus, is of importance to fields of marine ecology, chemistry, optics and any subject matter that involves systems with high optical depths and sensitivity.

References: