Introduction to Koopman Operator Theory for Dynamical Systems

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Advanced Mechanics Presentation, Fall 2021

Outline

1. History and Significance - *Koopmania*
2. Classical Approach to Dynamical Systems
3. Data Driven Approach and the **Koopman Operator**
4. Koopman Linear Expansion (KLE) and the Duffing Oscillator
5. For the *Culture*

History

The Koopman operator formalism originated in the early work of **Bernard Koopman** in 1931[1] inspired by Quantum Mechanics

- It is a classical analog to the quantum evolution operator

[1] https://www.pnas.org/content/pnas/17/5/315.full.pdf

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- It is a classical analog to the quantum evolution operator
- This work inspired John von Neumann's *Mean Ergodic Theorem*
- It provided an alternative formalism for study of dynamical systems

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Significance

The use of this operator became very popular in the latter part of the 20^{th} century. Applications include

- Model reduction and fault detection in energy systems for buildings,
- Stability assessment in power networks
- Extracting spatio-temporal patterns of brain activity
- Background detection and object tracking in videos
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Your Thoughts

Okay, we get that it is a fancy important operator but what exactly is it? How does it work?

- You

Dynamical Systems

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$$x^{t+1} = T(x^t) \mid x \in S \subset \mathbb{R}^n, t \in \mathbb{Z}, T : S \to S \tag{2}$$

Classical Approach - *Geometric Viewpoint*

- Historically dynamic systems were studied/developed using a geometrical viewpoint
- Phenomena in dynamic systems analyzed through tools such as *flow, limit cycles, equilibria, stability, invariant sets, attractors, bifurcation*

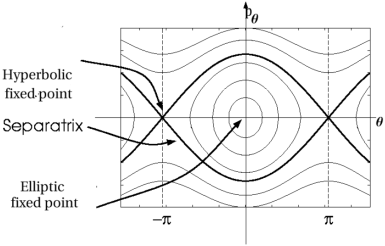

Figure 1: A Phase Space Plot[2]

[2] https://www.cmi.ac.in/~debangshu/dynamics.pdf

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2. **Try:** Find analytical solutions and use them to analyze the dynamics (i.e. find the attractors, invariant sets, imminent bifurcations, etc)

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1. Construct a model for the system in the form of (1) or (2).

3. **Try:** Use approximation techniques to evaluate the qualitative behavior of the system (e.g. construct Lyapunov functions to prove the stability of a fixed point)

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4. **Try:** Employ numerical computation and then extract information from a single or multiple simulated trajectories of the system

Classical Approach - *Algorithmic Summary*

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5. **Try:** Give up and cry

Motivation for an Alternative Approach

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- Cannot leverage this age of *Big Data*

Koopman Operator Formalism

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  - data is evaluation of functions of the state
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Consider the unforced motion of an incompressible fluid inside a box

- *state space* $\rightarrow$ set of all smooth velocity fields on the flow domain that satisfy the incompressibility condition
- *rule of evolution* $\rightarrow$ the Euler-Lagrange equations
- *observables* $\rightarrow$ pressure/vorticity at a given point in the flow domain, velocity at a set of points
- *data* $\rightarrow$ pressure and velocity sensor outputs

Motivating Question

Given the knowledge of an observable in the form of time series generated by experiment or simulation, what can we say about the evolution of the state?

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- Lifts the dynamics from the state space to the space of observables
  - Finds a coordinate transform where the dynamics are linear

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Linearity of U follows from the linearity of the composition operation

$$U[g_1 + g_2](x) = [g_1 + g_2] \circ T(x) = g_1 \circ T(x) + g_2 \circ T(x) = Ug_1(x) + Ug_2(x)$$

Koopman Operator

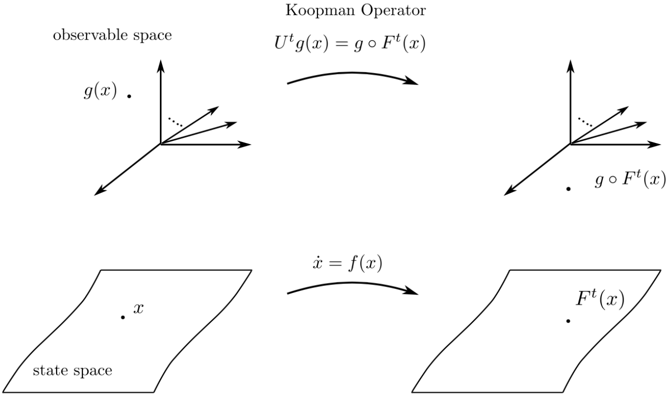

Figure 2: Visualization of the Koopman Operator acting on the state space[3]

USC

[3] https://arxiv.org/abs/2010.05377

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Thus, the evolution of observables

$$U^t g(x) = \sum_{k=0}^{\infty} g_k e^{\lambda_j t} \phi_k(x) \tag{6}$$

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Bad: Inifinite sum

Something Fishy?

- Assumption made in deriving KLE holds for a large class of nonlinear systems
  - e.g. hyperbolic fixed points, limit cycles and tori as attractors
  - The spectrum of U consists of only eigenvalues
  - The associated eigenfunctions span the space of observables
- Assumption does not hold for the class of chaotic dynamical systems

Duffing Oscillator

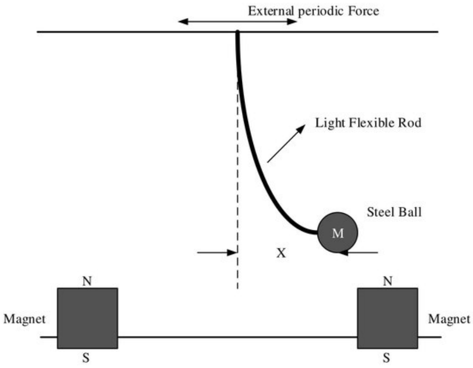

Figure 3: Duffing oscillator model[4]

[4] https://ieeexplore.ieee.org/document/8896260

Duffing Oscillator

Consider the nonlinear Duffing system - *a particle in a double potential well*

$$\ddot{x} = x - x^3 \tag{8}$$

with state space representation

$$\dot{x_1} = x_2 \mid \dot{x_2} = x_1 - x_1^3 \tag{9}$$

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- Three fixed points
  - a saddle at the origin $\rightarrow \lambda = \pm 1$
  - two centers at $(\pm 1, 0) \rightarrow \lambda = \pm\sqrt{2i}$
- local linearizations (shaded regions) around fixed points

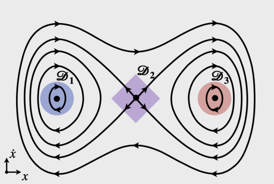

Figure 4: The Phase Space Plot[5]

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U on the Duffing Oscillator Problem

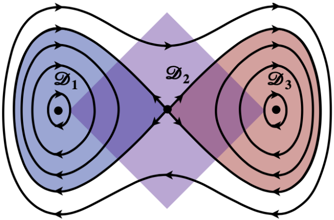

Figure 5: Koopman Linearization Domains[5]

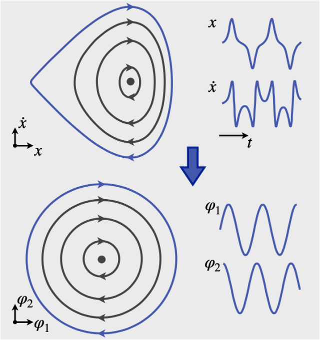

Figure 6: Koopman Coordinate Transform[5]

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- Koopman Continuous Spectrum (KCS)
  - extends Koopman operator theory to model Chaotic systems
- Dynamic Mode Decomposition (DMD) Algorithms
  - teaches computers physics (our jobs are in danger)
  - `https://www.youtube.com/watch?v=Kap3TZwAsv0&list=PLMrJAkhIeNNR6DzT17-MM1GHLkuYVjhyt`

Questions?